Exercise 10

Solve the differential equation or initial-value problem using the method of undetermined coefficients.

$$y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1, \quad y'(0) = 0$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + y_c' - 2y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} + re^{rx} - 2(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + r - 2 = 0$$

Solve for r.

$$(r+2)(r-1) = 0$$

$$r = \{-2, 1\}$$

Two solutions to the ODE are e^{-2x} and e^x. By the principle of superposition, then,

$$y_c(x) = C_1 e^{-2x} + C_2 e^x.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' + y_p' - 2y_p = x + \sin 2x \tag{2}$$

Since the inhomogeneous term is the sum of a polynomial and a sine function, the particular solution is $y_p = (A + Bx) + (C \cos 2x + D \sin 2x)$.

$$y_p = (A + Bx) + (C \cos 2x + D \sin 2x) \quad \rightarrow \quad y_p' = (B) + (-2C \sin 2x + 2D \cos 2x)$$

$$\rightarrow \quad y_p'' = (-4C \cos 2x - 4D \sin 2x)$$

Substitute these formulas into equation (2).

$$(-4C \cos 2x - 4D \sin 2x) + [(B) + (-2C \sin 2x + 2D \cos 2x)] - 2[(A + Bx) + (C \cos 2x + D \sin 2x)] = x + \sin 2x$$

$$(B - 2A) + (-2B)x + (-4C - 2C + 2D) \cos 2x + (-4D - 2C - 2D) \sin 2x = x + \sin 2x$$

Match the coefficients on both sides to get a system of equations for A, B, C, and D.

$$\left.\begin{aligned} B - 2A &= 0 \\ -2B &= 1 \\ -4C - 2C + 2D &= 0 \\ -4D - 2C - 2D &= 1 \end{aligned}\right\}$$

Solving it yields

$$A = -\frac{1}{4} \quad \text{and} \quad B = -\frac{1}{2} \quad \text{and} \quad C = -\frac{1}{20} \quad \text{and} \quad D = -\frac{3}{20},$$

which means the particular solution is

$$y_p = -\frac{1}{4} - \frac{1}{2}x - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1 e^{-2x} + C_2 e^{x} - \frac{1}{4} - \frac{1}{2}x - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x, \end{aligned}$$

where C_1 and C_2 are arbitrary constants. Differentiate it with respect to x.

$$y'(x) = -2C_1 e^{-2x} + C_2 e^{x} - \frac{1}{2} + \frac{1}{10}\sin 2x - \frac{3}{10}\cos 2x$$

Apply the boundary conditions to determine C_1 and C_2.

$$y(0) = C_1 + C_2 - \frac{3}{10} = 1$$

$$y'(0) = -2C_1 + C_2 - \frac{4}{5} = 0$$

Solving the system yields $C_1 = 1/6$ and $C_2 = 17/15$. Therefore,

$$y(x) = \frac{1}{6}e^{-2x} + \frac{17}{15}e^{x} - \frac{1}{4} - \frac{1}{2}x - \frac{1}{20}\cos 2x - \frac{3}{20}\sin 2x.$$